Nonlinear Generation of Vorticity by Surface Waves

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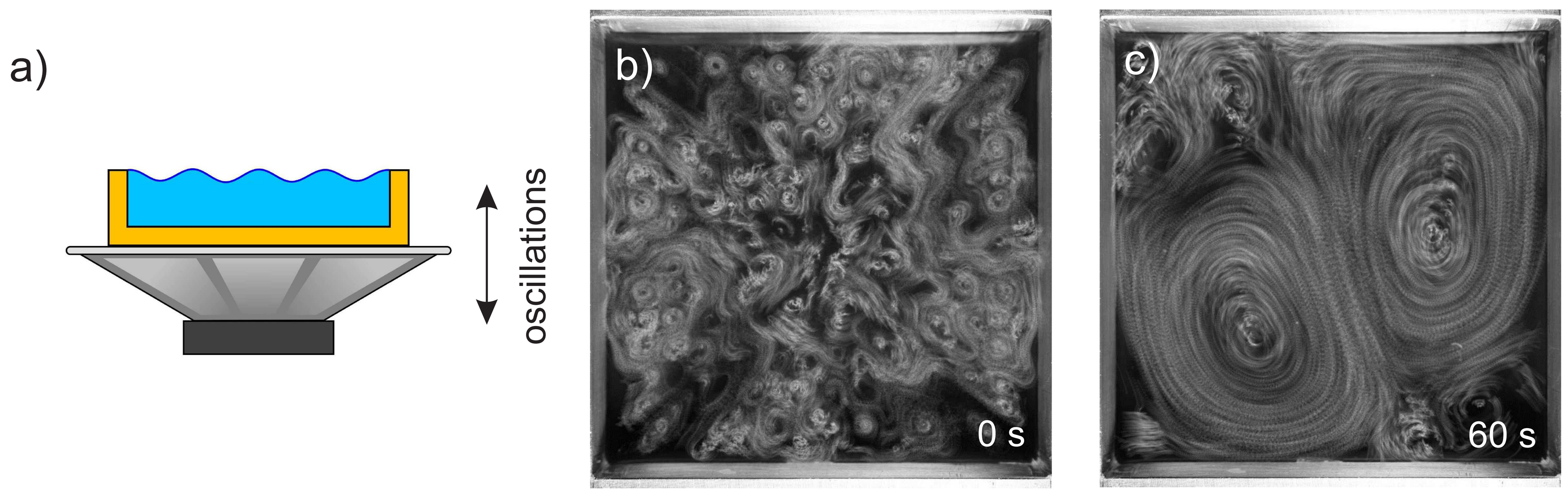
1. Motivation

Recently, solenoidal currents generated by surface waves were studied experimentally [1-3]. It was shown that statistical properties of horizontal motion are similar to statistics of 2d turbulence, despite the fact that velocity field was substantially 3d. There is **no explanation** to the observation, even the generation mechanism of solenoidal currents from surface waves **remained** so far **obscure**. Here we present a first-ever theory explaining the generation mechanism and check it experimentally.

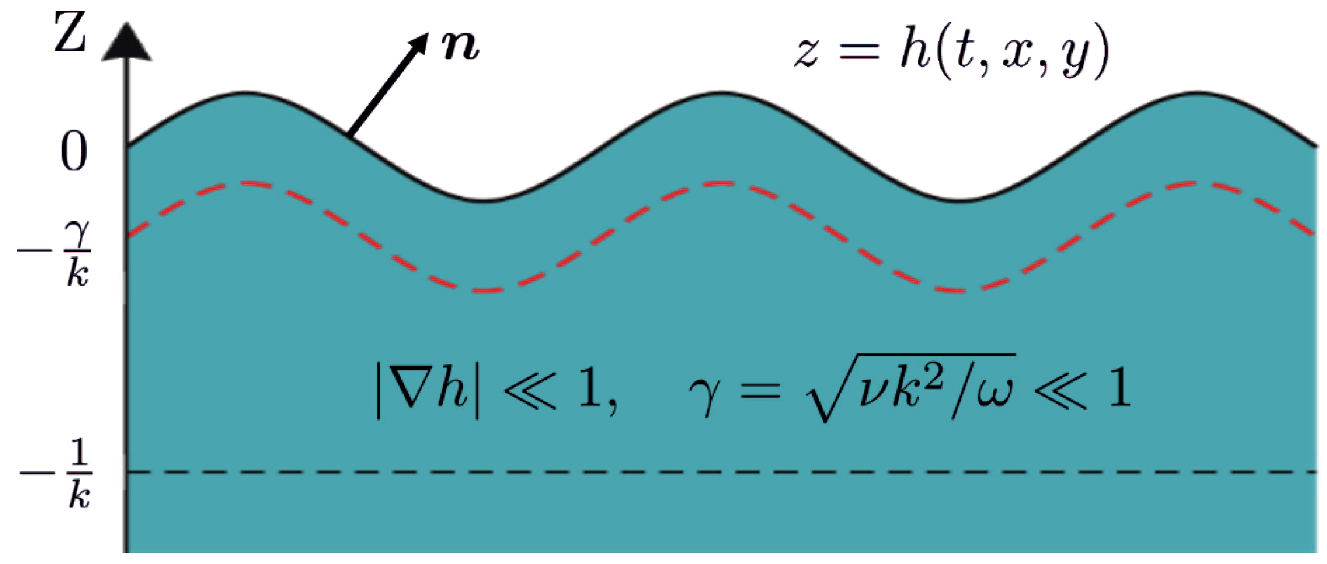


(a) Experimental setup, (b) regular pattern of interacting vortices, (c) statistically stationary distribution of vortices.

What is our aim?

- Some surface waves are excited on a fluid surface, we know the elevation $h(t,x,y)$.
- We want to find vertical component of the vorticity, based on the Navier-Stokes equation.

2. Linear Approximation & Qualitative Explanation



The vorticity is **directed along the surface** and concentrated in a thin viscous sublayer near the fluid surface:

$$\varpi_\alpha = 2\epsilon_{\alpha\beta}\exp(\hat{\kappa}z)\partial_\beta\partial_t h + O(\gamma^2), \quad \hat{\kappa}^2 = \partial_t/\nu + \hat{k}^2.$$

To find the **vertical vorticity** we should take into account the main nonlinear contribution:

One can roughly say that the surface tilt produces a tilt of the vorticity in the viscous sublayer as well.

Since the horizontal vorticity is independent of viscosity on a fluid surface, the vertical vorticity will be also **independent of viscosity** though it is produced by the viscous mechanism.

3. Nonlinear Vorticity Generation

Up to the **second-order** the vertical vorticity obeys the following equation:

$$\partial_t\varpi_z - \nu\nabla^2\varpi_z = \varpi_\alpha\partial_\alpha v_z.$$

The right-hand side corresponds to the rotation of horizontal vorticity by the velocity field. The boundary condition posed at $z=0$:

$$\partial_z\varpi_z = \partial_\alpha h\partial_z\varpi_\alpha - \epsilon_{\alpha\gamma}(\partial_\alpha v_\beta + \partial_\beta v_\alpha)\partial_\beta\partial_\gamma h.$$

Monochromatic pumping:

$$\varpi_z(z) = 2\epsilon_{\alpha\beta}(e^{\hat{\kappa}z}\partial_\beta\partial_t h)(e^{\hat{k}z}\partial_\alpha h) + 2\epsilon_{\alpha\beta}\hat{k}^{-1}e^{\hat{k}z}(\partial_\alpha h\partial_\beta\partial_t\hat{k}h + \partial_\alpha\partial_\gamma h\partial_\beta\partial_\gamma\partial_t\hat{k}^{-1}h).$$

First term — tilt of ϖ_α due to $\partial_\alpha h$ (penetrates on a distance γ/k). Second term — spreading of rotated vorticity into the bulk. Last term is related to the non-zero curvature of the surface (penetrates on a distance $1/k$).

The **theory is correct** if higher-order nonlinear terms are small compared to the kept ones $\Leftrightarrow kh \ll \gamma$.

4. Stokes Drift

We restore the velocity field analyzing the motion of tracers located at fluid surface. The equation of motion is

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{v}(\boldsymbol{X}, t),$$

and near some point $\boldsymbol{x}_0$ we can expand a velocity field in Taylor series. Up to the second-order we obtain

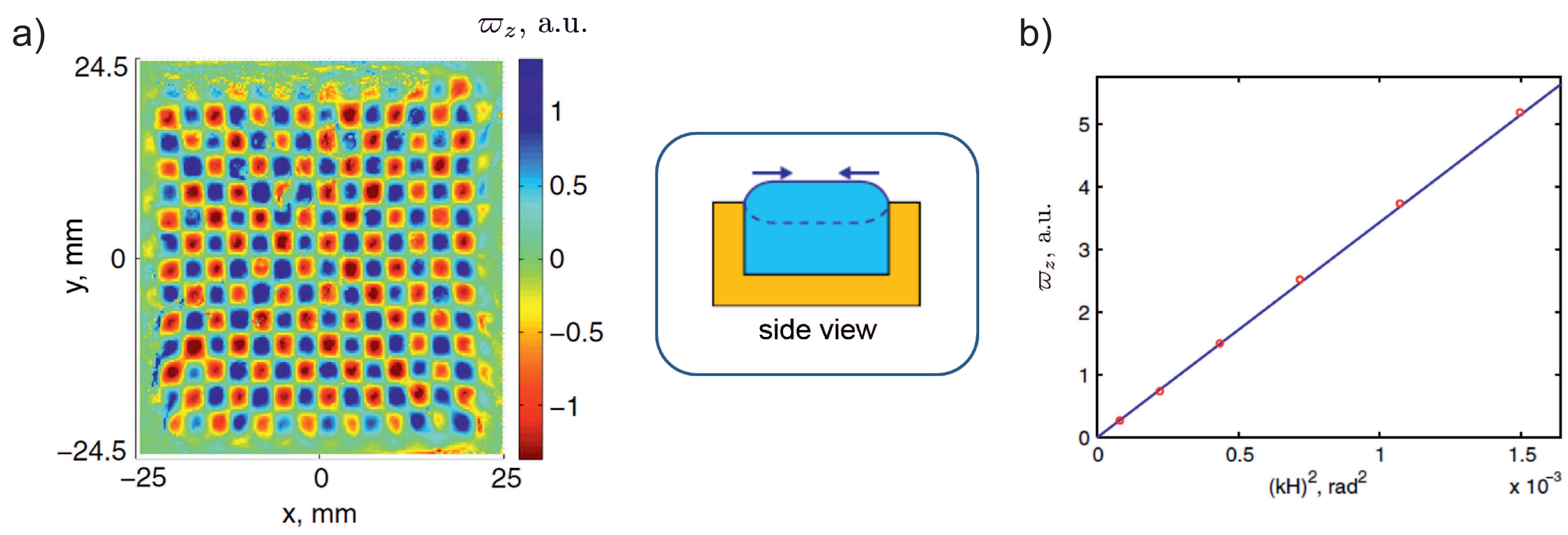
$$\delta\boldsymbol{X}_0 = \int \boldsymbol{v}(\boldsymbol{x}_0, h(\boldsymbol{x}_0, t))dt, \ \delta\boldsymbol{X}_1 = \int \hat{G}\cdot\delta\boldsymbol{X}_0 dt,$$

where $\hat{G}$ is a velocity gradient tensor.

Measured vorticity for orthogonal plane waves:

$$\varpi_L = \epsilon_{\alpha\beta}\partial_\alpha v_\beta(\boldsymbol{x}, h(\boldsymbol{x}, t)) = \varpi_z(0) + \epsilon_{\alpha\beta}\partial_\alpha h\partial_z v_\beta(\boldsymbol{x}, 0).$$

5. Nearly Square Cell with Symmetrical Walls



(a) The waves are excited due to **water meniscus** formed near the walls. The surface elevation can be modeled as

$$h = H_1\cos(\omega t)\cos(kx) + H_2\cos(\omega t + \psi)\cos(ky),$$

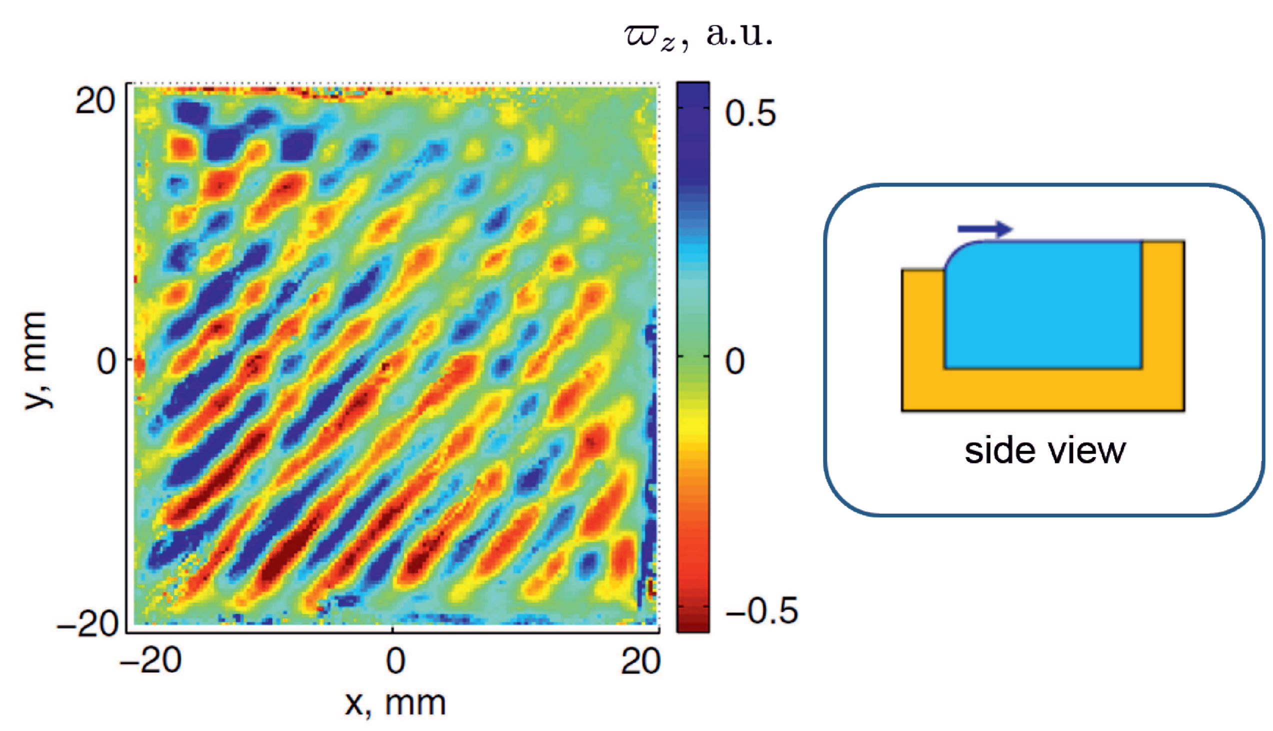
the phase shift ψ is related to the cell asymmetry. For the **measured vorticity** we obtain

$$\varpi_L = -(1+\sqrt{2})\sin\psi H_1 H_2\omega k^2\sin(kx)\sin(ky).$$

The **Stokes drift** correction has the same spatial structure as $\varpi_z(0)$, but with the prefactor $2+\sqrt{2}$ replaced by -1. Their sum gives the answer which is written above.

(b) Vorticity for different pumping amplitudes plotted as a function of the tilt amplitude kH. The line corresponds to the dependence $\varpi_z \propto (kH)^2$ and it proves that the vorticity is generated by the second-order nonlinearity.

6. Square Cell with Asymmetrical Walls



Left and bottom walls are slightly lower than others. The water level is adjusted to produce mainly two running waves from the lower walls. The surface elevation can be modeled as

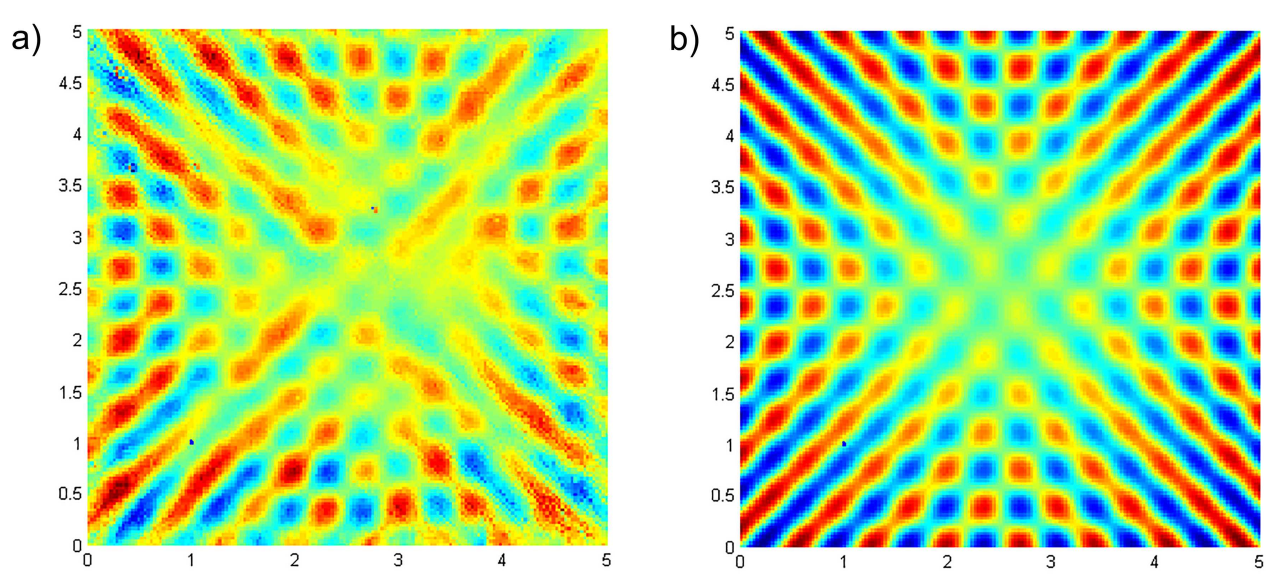
$$h = H_1\cos(\omega t - kx) + H_2\cos(\omega t - ky)$$

and for the **measured vorticity** we obtain

$$\varpi_L = -(1+\sqrt{2})H_1 H_2\omega k^2\sin(kx - ky).$$

The direction of particles' motion cannot be explained by the **Stokes drift alone**, since this mechanism moves particles in the opposite direction.

7. Square Cell with Symmetrical Walls: Wave Damping



Vorticity ϖ_L in a perfectly square cell where the phase shift $\psi \ll 1$ due to the **cell symmetry:** (a) experiment and (b) theory.

Taking into account the **wave damping** we can model the surface elevation as

$$h = \frac{H_1}{2}\Big[\cos(kx-\omega t)e^{-\alpha x} + \cos(kx+\omega t)e^{\alpha x}\Big] + \frac{H_2}{2}\Big[\cos(ky-\omega t)e^{-\alpha y} + \cos(ky+\omega t)e^{\alpha y}\Big]$$

and for the **measured vorticity** we obtain

$$\varpi_L = \frac{(1+\sqrt{2})}{2}H_1 H_2\omega k^2\Big[\sin(kx-ky)\sinh(\alpha x + \alpha y) - \sin(kx+ky)\sinh(\alpha x - \alpha y)\Big].$$

8. Conclusion

- Suggested theory can be used to analyze the solenoidal motion on the **ocean surface**.
- **Practical Application:** the theory can be used to design solenoidal currents (e.g., for mixing problems).
- Our results allow to understand better the phenomenon of surface turbulence driven by Faraday waves.

9. References

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